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THE LAMINAR HYPERSONIC TRAIL DOWNSTREAM OF A LIFT AIRFOIL

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The asymptotics of solution of the Navier — Stokes equation which determines the flow at considerable distance downstream of a lift airfoil of finite dimensions is investigated. The velocity field is divided in two regions. In the outer region the motion of gas conforms to Euler's equation, while the inner region contains a laminar trail which is determined in the longitudinal direction by the heat flux and by tangential viscous stresses. The continuation of solution from one region to the other is achieved by the method of joining external and internal asymptotic expansions. In the case of three-dimensional flows the problem of joining is complicated by the oscillatory character of the trail external boundary induced by the lift force.

1. Let us consider a steady hypersonic flow at considerable distance from a lift airfoil subjected to drag and lift. We denote the density of gas in the oncoming stream by ρ_∞ and its velocity along the axis of a cylindrical system of coordinates x, r, φ by v_∞. We assume that upstream of the bow shock wave the pressure $p_\infty = 0$ and the Mach number $M_\infty = \infty$, and that the gas conforms to the Clapeyron equation of state, with both specific heats c_p and c_v constant and their ratio denoted by $\varkappa$, assumed to be $1 < \varkappa < 2$. The dependence of viscosity coefficients λ_1 and λ_2 and of thermal conductivity k on specific enthalpy w is assumed linear: $\lambda_1 = \lambda_{10} w$, $\lambda_2 = \lambda_{20} w$, and $k = k_0 w$. We introduce the Prandtl number $N_{Pr} = c_p \lambda_{10} / k_0$. It is convenient to use dimensionless variables and unknown functions taking ρ_∞, v_∞, and λ_{10} as the fundamental reference units.

The principal terms of the asymptotic solution of the Navier — Stokes equations at considerable distance downstream of a finite body in an axisymmetric hypersonic flow were determined by Sychev [1]. The stream pattern derived by him consists of two essentially different regions. In establishing the form of asymptotics for the outer region it is possible to neglect the effects of viscosity and thermal conductivity, where the motion of gas obeys the simpler Euler's equations. The outer region is separated from the oncoming stream by a shock wave whose structure was investigated in [2]. The axisymmetric velocity field is defined by the solution of the problem of strong fuse explosion indicated by Sedov [3,4]. In that solution the dimensionless parameters of gas depend on the single self-similar variable

$$\xi = \frac{r}{(bx)^{1/2}}, \quad b = \mathrm{const} \tag{1.1}$$

In conformity with the described results of axisymmetric flow analysis we shall distinguish in the solution of the three-dimensional problem two regions (Fig. 1), and neglect in the outer region the effects of viscosity and thermal conductivity on the motion of gas. The principal terms of the asymptotic solution of Euler's equation under such conditions are presented in [5].

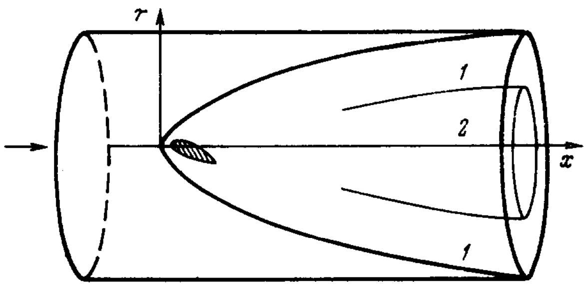

Fig. 1

If in addition to drag the streamlined body is subjected to lift, the shape of the shock wave for $x \to \infty$ is specified by formula

$$r_s = (bx)^{1/2} (1 + b_y x^{-1/2} \ln x \cos \varphi + \ldots) \tag{1.2}$$

where the constant b_y is proportional to the lift force F_y.

It will be readily seen that all terms of the solution of Euler 's equation corresponding to the compression shock (1.2) simultaneously represent the asymptotic solution of the Navier — Stokes input equations. The systems of ordinary differential equations for functions of the self-similar variable ξ are the same independently of whether the coefficients of viscosity and thermal conductivity are assumed zero or to have finite values. For plane-parallel flows around a lift profile this problem was considered in detail in [6].

The inner region indicated by the numeral 2 in Fig. 1 contains the trailing vortex. In that region it is no longer possible to neglect the viscosity and thermal conductivity of gas, since the solution that corresponds to the shock wave (1.2) the terms rejected in the Navier — Stokes equations begin to increase rapidly [5]. Sychev [1] introduced in the analysis of flow in the trail the variable

$$\zeta = \xi x^{(\varkappa-1)/2(\varkappa+1)} = \frac{r}{b^{1/2} x^{1/(\varkappa+1)}} \tag{1.3}$$

The form of solution in the trail is determined by the behavior of gasdynamic functions for $\xi \to 0$. We denote the projections of the velocity vector on the axes x, r, and φ by v_x, v_r, and v_φ. Expressing the asymptotic formulas presented in [5, 7] in terms of the variable ζ, we conclude that

$$v_x = 1 - \frac{1}{2(\varkappa+1)} b x^{-\varkappa/(\varkappa+1)} [v_{x21}(\zeta) + F_{v_x}(x, \zeta) \cos \varphi + \ldots] \tag{1.4}$$

$$v_r = \frac{1}{\varkappa+1} b^{1/2} x^{-\varkappa/(\varkappa+1)} [v_{r21}(\zeta) + F_{v_r}(x, \zeta) \cos \varphi + \ldots]$$

$$v_\varphi = \frac{1}{\varkappa+1} b^{1/2} F_{v_\varphi}(x, \zeta) \sin \varphi + \ldots$$

$$\rho = \frac{\varkappa+1}{\varkappa-1} x^{-1/(\varkappa+1)} \{\rho_{21}(\zeta) + F_\rho(x, \zeta) \cos \varphi + \ldots\}$$

$$p = \frac{1}{2(\varkappa+1)} \frac{b}{x} \{p_{21}(\zeta) + x^{-\varkappa/(\varkappa+1)} [p_{22}(\zeta) + F_p(x, \zeta) \cos \varphi + \ldots]\} \qquad (1.4)$$

$$w = \frac{\varkappa}{2(\varkappa+1)^2} b x^{-\varkappa/(\varkappa+1)} \{w_{21}(\zeta) + F_w(x, \zeta) \cos \varphi + \ldots\}$$

$$F_q = b_y x^{-k_4} [q_{2c}(\zeta) \cos (k_3 \ln x) + q_{2s}(\zeta) \sin (k_3 \ln x)]$$

$$q = v_x,\ v_r,\ v_\varphi,\ \rho,\ p,\ w.$$

$$k_3 = \frac{\varkappa-1}{2(\varkappa+1)} \sqrt{\frac{3-\varkappa}{\varkappa-1}}, \quad k_4 = \frac{2-\varkappa}{2(\varkappa+1)}$$

Functions with subscript 21, which form the first approximation, determine the trail structure downstream of any body subjected only to drag. To derive boundary conditions for these it is sufficient to know only the principal terms of the asymptotics that establish the distribution of parameters of air in the vicinity of the fuse explosion. Let us consider the intermediate region where $r = \eta x^{\alpha}$, $1 / (\varkappa + 1) < \alpha < 1/2$. The passing to limit $x \to \infty$ in that region is effected with $\eta = \mathrm{const}$, when, as implied by formulas (1.1) and (1.3), $\zeta \to \infty$ and $\xi \to 0$. In conformity with the method of joining outer and inner asymptotic expansions we have [8, 9]

$$v_{x21} = \frac{\varkappa}{\varkappa+1} \frac{k_2}{k_1} \zeta^{-2/(\varkappa-1)} + \ldots, \quad v_{r21} = \frac{\varkappa+1}{2\varkappa} \zeta + \ldots \qquad (1.5)$$

$$\rho_{21} = k_1 \zeta^{2/(\varkappa-1)} + \ldots, \quad p_{21} = k_2 + \ldots, \quad w_{21} = \frac{k_2}{k_1} \zeta^{-2/(\varkappa-1)} + \ldots$$

in which the dependence of coefficients k_1 and k_2 on $\varkappa$ is given in [10].

Function p_{22} is generated by the second term of the asymptotic expansion for the pressure in the solution of the problem of strong explosion. The boundary condition for it states that

$$p_{22} = \frac{(\varkappa+1)^2 (\varkappa-1)}{4\varkappa^3} k_1 \zeta^{2\varkappa/(\varkappa-1)} + \ldots \qquad (1.6)$$

Finally, for the functions that depend on the asymptotic perturbations of the stress because of the lift acting on the body with $\zeta \to \infty$, we obtain

$$v_{x2c} = -\frac{\varkappa}{\varkappa+1} \frac{k_2}{k_1^2} \zeta^{-(2\varkappa+1)/(\varkappa-1)} [c_2 \cos (k \ln \zeta) + c_3 \sin (k \ln \zeta)] + \ldots \qquad (1.7)$$

$$v_{r2c} = \frac{(\varkappa+1)(\varkappa-1)^2}{4\varkappa k_1} \zeta^{-\varkappa/(\varkappa-1)} [(-c_2 + k c_3) \cos (k \ln \zeta) -$$
$$(k c_2 + c_3) \sin (k \ln \zeta)] + \ldots$$

$$v_{\varphi 2c} = \frac{\varkappa^2 - 1}{4\varkappa k_1} \zeta^{-\varkappa/(\varkappa-1)} \{[(2-\varkappa) c_2 + k\varkappa c_3] \cos (k \ln \zeta) +$$
$$[-k\varkappa c_2 + (2-\varkappa) c_3] \sin (k \ln \zeta)\} + \ldots$$

$$\rho_{2c} = \zeta^{(3-2\varkappa)/(\varkappa-1)} [c_2 \cos (k \ln \zeta) + c_3 \sin (k \ln \zeta)] + \ldots$$

$$p_{2c} = \frac{(\varkappa+1)^2 (\varkappa-1)}{2\varkappa^2} \zeta^{1/(\varkappa-1)} \left[\left(\frac{1}{\varkappa-1} c_2 - k c_3\right) \cos (k \ln \zeta) +\right.$$
$$\left.\left(k c_2 + \frac{1}{\varkappa-1} c_3\right) \sin (k \ln \zeta)\right] + \ldots$$

$$w_{2c} = -\frac{k_2}{k_1^2}\zeta^{-(2\varkappa+1)/(\varkappa-1)}\left[c_2\cos(k\ln\zeta) + c_3\sin(k\ln\zeta)\right] + \ldots$$
$$k = \sqrt{(3-\varkappa)/(\varkappa-1)}$$

The expressions for $v_{x2s}, v_{r2s}, \ldots, w_{2s}$ not shown above are obtained from related expressions for $v_{x2c}, v_{r2c}, \ldots, w_{2c}$ by the substitution $c_2 \to -c_3$ and $c_3 \to c_2$. Constants c_2 and c_3 are obtained from the solution of the Cauchy problem for functions of the self-similar variable ξ in the outer region [5]. For $\varkappa = 1.4$ they are $c_2 = 0.085$ and $c_3 = -0.978$.

2. The substitution of expansions (1.4) into the input Navier — Stokes equations yields a system of ordinary differential equations for first approximation functions

$$\rho_{21}\frac{dv_{r21}}{d\zeta} + (v_{r21}-\zeta)\frac{d\rho_{21}}{d\zeta} + \left(\frac{v_{r21}}{\zeta} - 1\right)\rho_{21} = 0 \qquad (2.1)$$
$$\frac{k_5}{N_{Pr}}w_{21}\frac{d^2w_{21}}{d\zeta^2} + \left[\frac{k_5}{N_{Pr}}\left(\frac{dw_{21}}{d\zeta} + \frac{w_{21}}{\zeta}\right) - \rho_{21}(v_{r21}-\zeta)\right]\frac{dw_{21}}{d\zeta} +$$
$$\varkappa\rho_{21}w_{21} = \frac{\varkappa^2-1}{\varkappa}p_{21}, \quad \frac{dp_{21}}{d\zeta} = 0, \quad p_{21} = \rho_{21}w_{21}$$
$$k_5w_{21}\frac{d^2v_{x21}}{d\zeta^2} + \left[k_5\left(\frac{dw_{21}}{d\zeta} + \frac{w_{21}}{\zeta}\right) - \rho_{21}(v_{r21}-\zeta)\right]\frac{dv_{x21}}{d\zeta} +$$
$$\varkappa\rho_{21}v_{x21} = (\varkappa-1)p_{21}, \quad k_5 = \frac{\varkappa(\varkappa-1)}{2(\varkappa+1)^2}$$

The sought solution must satisfy the limit relationships (1.5) for $\zeta \to \infty$. When $\zeta = 0$ the conditions of flow symmetry together with the stipulation of absence of heat sources yield $v_{r21} = dv_{x21}/d\zeta = dw_{21}/d\zeta = 0$.

In system (2.1) the equation for the longitudinal component v_{x21} of the velocity vector differs from the remaining ones which depend only on those terms of the Navier — Stokes equations which are related to heat transfer, while viscous properties of gas have no effect on the form of these. On the other hand, in the derivation of the equation for v_{x21} the allowance for viscosity is essential. When $N_{Pr} = 1$ system (2.1) is simpler, since the left-hand sides of the second and fifth equations are then the same [1].

Let $N_{Pr} \neq 1$. System (2.1) without the equation for v_{x21} can then be integrated twice. Carrying it out and using the conditions of symmetry for $\zeta = 0$, we obtain for function w_{21} a nonlinear differential equation of the second order. The stipulation of absence of heat sources makes it possible to state that in the neighborhood of zero

$$w_{21} = c_4 + c_5\zeta^2 + \ldots, \quad c_5 = -\frac{(\varkappa+1)^2k_2}{2\varkappa^2(\varkappa-1)}\frac{N_{Pr}}{c_4} \qquad (2.2)$$

where c_4 is an arbitrary constant. Other constants are denoted below by the same letter with appropriate subscripts.

The behavior of function w_{21} when $\zeta \to \infty$ is determined by the asymptotic formula

$$w_{21} = \frac{k_2}{k_1}\zeta^{-2/(\varkappa-1)}\left[1 + O(\zeta^{k_6})\right] + w_{21}^\circ(\zeta, c_6), \quad k_6 = -\frac{2(\varkappa+1)}{\varkappa-1} \qquad (2.3)$$

which contains the exponentially small quantity $w_{21}^{\circ}(\zeta, c_6)$.

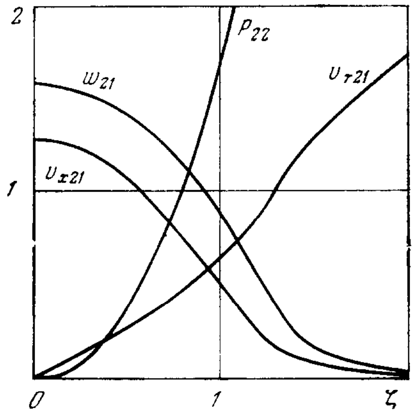

Fig. 2

The boundary value problem can now be formulated as follows: find for w_{21} a solution which in the zero neighborhood satisfies formula (2.2) and for $\zeta \to \infty$ satisfies the asymptotic formula (2.3). The results of numerical solution of the problem with $\varkappa = 1.4$ and $N_{Pr} = 0.75$ are shown in Fig. 2. The constant determined in the course of solution is $c_4 = 1.571$.

The boundary value problem for v_{x21} is derived similarly. The condition of symmetry for $\zeta = 0$ imples that

$$v_{x21} = \frac{\varkappa - 1}{\varkappa} c_4 + c_7 \left[1 - \frac{(\varkappa+1)^2 k_2}{\varkappa - 1} \frac{1}{c_4^2} \zeta^2 + \ldots\right] \tag{2.4}$$

The asymptotic formula for v_{x21} when $\zeta \to \infty$ is of the form

$$v_{x21} = \frac{\varkappa}{\varkappa+1} \frac{k_2}{k_1} \zeta^{-2/(\varkappa-1)} + \ldots + c_8 (\zeta^{-2\varkappa^2/(\varkappa-1)} + \ldots) + \overset{\circ}{v}_{x21}(\zeta, c_9) \tag{2.5}$$

with the exponentially fading term $\overset{\circ}{v}_{x21}(\zeta, c_9)$ in the right-hand side. To determine constant c_8 it is necessary to add supplementary terms in the expansion of solution for the outer region where the motion of gas conforms to Euler 's equations. Using the properties of functions appearing in the supplementary terms [11], we conclude that $c_8 = 0$. The curve of function v_{x21} specified in the zero neighborhood by formula (2.4) and approximated by formula (2.5) when $\zeta \to \infty$ and $c_8 = 0$ is shown in Fig. 2, with the calculated constant $c_7 = 0.830$.

Function p_{22} satisfies the simplest differential equation

$$\frac{dp_{22}}{d\zeta} = \frac{\varkappa}{(\varkappa+1)^2} \left\{\mu_1 \left[w_{21} \frac{d^2 v_{r21}}{d\zeta^2} + \left(\frac{dw_{21}}{d\zeta} + \frac{w_{21}}{\zeta}\right) \frac{dv_{r21}}{d\zeta} - \frac{w_{21} v_{r21}}{\zeta^2}\right] + \mu_2 \frac{v_{r21}}{\zeta} \frac{dw_{21}}{d\zeta}\right\} - \frac{2}{\varkappa - 1} \rho_{21} \left[(v_{r21} - \zeta) \frac{dv_{r21}}{d\zeta} - \varkappa v_{r21}\right]$$

$$\mu_1 = {}^4/_3 + \lambda_{20} / \lambda_{10}, \quad \mu_2 = -{}^2/_3 + \lambda_{20} / \lambda_{10}$$

Let us specify that function p_{22} must remain regular when $\zeta \to 0$. The boundary condition (1.6) is satisfied when $\zeta \to \infty$, but does not provide the possibility of determining the additive constant c_{10}. The latter can be determined by considering the axisymmetric problem of external approximations in the outer region. For the subsequent analysis it is important that functions related to the asymmetry of flow are independent of c_{10}. The curve of p_{22} is shown in Fig. 2 for $c_{10} = 0$.

3. We pass to the analysis of perturbations induced in the trail by the lift of the streamlined body. We surround the latter by a control surface (Fig. 1) and determine

the momentum component in the projection on the y axis, which is transferred through the part of the plane $x = \mathrm{const}$ within the trail boundary. Expansions (1.4) show that when $x \to \infty$ the integral defining that component is vanishingly small, which in other words means that the lift can be determined by the outer flow parameters.

It was shown in [5] that the distribution of gas parameters acquire an oscillatory character when approaching the inner boundary of that region, and the amplitude of the perturbed velocity vector components increases indefinitely. For $1.5 < \varkappa < 2$ the amplitude of excess density oscillations also increases indefinitely. It is interesting that oscillations begin to develop in the inviscid outer part of the stream, where inertia forces are balanced only by pressure forces. The oscillations generated at the trail boundary continue over its whole length. This results in the appearance in formulas (1.4), which determine the asymptotic form of solution in the trail downstream of the body, of terms $\cos(k_3 \ln x)$ and $\sin(k_3 \ln x)$. Behavior of the stream near the outer boundary of the inner region is defined by formulas (1.7) that contain terms $\cos(k \ln \zeta)$ and $\sin(k \ln \zeta)$. Variation of oscillation frequencies along and across the wave is, thus different.

Substituting expansions (1.4) into the Navier – Stokes equations, we obtain a homogeneous linear system of ordinary differential equations for second approximation functions. Analysis of the general properties of that system is considerably simplified by the introduction of the complex quantities $v_x^* = v_{x2c} + iv_{x2s}, \ldots, w^* = w_{2c} + iw_{2s}$ As a result, we obtain

$$\rho_{21}\frac{dv_r^*}{d\zeta} + (v_{r21} - \zeta)\frac{d\rho^*}{d\zeta} = -\left(\frac{d\rho_{21}}{d\zeta} + \frac{\rho_{21}}{\zeta}\right)v_r^* - \frac{\rho_{21}}{\zeta}v_\varphi^* - \left(\frac{dv_{r21}}{d\zeta} + \frac{v_{r21}}{\zeta} - \frac{4-\varkappa}{2} - ik\frac{\varkappa-1}{2}\right)\rho^* \qquad (3.1)$$

$$\begin{aligned}
&k_5\mu_1 w_{21}\frac{d^2v_r^*}{d\zeta^2} + \left[k_5\mu_1\left(\frac{dw_{21}}{d\zeta} + \frac{w_{21}}{\zeta}\right) - \rho_{21}(v_{r21} - \zeta)\right]\frac{dv_r^*}{d\zeta} + \\
&\quad k_5\mu_3\frac{w_{21}}{\zeta}\frac{dv_\varphi^*}{d\zeta} + k_5\left(\mu_1\frac{dv_{r21}}{d\zeta} + \mu_2\frac{v_{r21}}{\zeta}\right)\frac{dw^*}{d\zeta} - \frac{\varkappa-1}{2}\frac{dp^*}{d\zeta} = \\
&\quad \left[k_5\left(-\mu_2\frac{1}{\zeta}\frac{dw_{21}}{d\zeta} + \mu_4\frac{w_{21}}{\zeta^2}\right) + \rho_{21}\left(\frac{dv_{r21}}{d\zeta} - \frac{2+\varkappa}{2} - \right.\right. \\
&\quad \left.\left. ik\frac{\varkappa-1}{2}\right)\right]v_r^* + k_5\left(-\mu_2\frac{1}{\zeta}\frac{dw_{21}}{d\zeta} + \mu_4\frac{w_{21}}{\zeta^2}\right)v_\varphi^* - \\
&\quad k_5\mu_1\left(\frac{d^2v_{r21}}{d\zeta^2} + \frac{1}{\zeta}\frac{dv_{r21}}{d\zeta} - \frac{v_{r21}}{\zeta^2}\right)w^* + \left[(v_{r21} - \zeta)\frac{dv_{r21}}{d\zeta} - \varkappa v_{r21}\right]\rho^*
\end{aligned}$$

$$\begin{aligned}
&k_5 w_{21}\frac{d^2v_\varphi^*}{d\zeta^2} - k_5\mu_3\frac{w_{21}}{\zeta}\frac{dv_r^*}{d\zeta} + \left[k_5\left(\frac{dw_{21}}{d\zeta} + \frac{w_{21}}{\zeta}\right) - \rho_{21}(v_{r21} - \zeta)\right]\frac{dv_\varphi^*}{d\zeta} = \\
&\quad k_5\left(\frac{1}{\zeta}\frac{dw_{21}}{d\zeta} + \mu_4\frac{w_{21}}{\zeta^2}\right)v_r^* + \left[k_5\left(\frac{1}{\zeta}\frac{dw_{21}}{d\zeta} + \mu_4\frac{w_{21}}{\zeta^2}\right) + \right. \\
&\quad \left.\rho_{21}\left(\frac{dv_{r21}}{d\zeta} - \frac{2+\varkappa}{2} - ik\frac{\varkappa-1}{2}\right)\right]v_\varphi^* + \\
&\quad k_5\frac{1}{\zeta}\left(\mu_2\frac{dv_{r21}}{d\zeta} + \mu_1\frac{v_{r21}}{\zeta}\right)w^* - \frac{\varkappa-1}{2}\frac{p^*}{\zeta}
\end{aligned}$$

$$\frac{k_5}{N_{Pr}}w_{21}\frac{d^2w^*}{d\zeta^2} + \left[\frac{k_5}{N_{Pr}}\left(2\frac{dw_{21}}{d\zeta} + \frac{w_{21}}{\zeta}\right) - \rho_{21}(v_{r21} - \zeta)\right]\frac{dw^*}{d\zeta} =$$

$$\rho_{21}\frac{dw_{21}}{d\zeta}v_r^* - \left[\frac{k_5}{N_{Pr}}\left(\frac{d^2w_{21}}{d\zeta^2}+\frac{1}{\zeta}\frac{dw_{21}}{d\zeta}-\frac{w_{21}}{\zeta^2}\right)+\right.$$
$$\left.\rho_{21}\left(\frac{2+\varkappa}{2}+ik\frac{\varkappa-1}{2}\right)\right]w^* + \left[(v_{r21}-\zeta)\frac{dw_{21}}{d\zeta}-\varkappa w_{21}\right]\rho^*$$
$$w_{21}\rho^* + \rho_{21}w^* = 0,\quad \mu_3 = {}^1/_3 + \lambda_{20}/\lambda_{10},\quad \mu_4 = {}^7/_3 + \lambda_{20}/\lambda_{10}$$

Note the basic difference between first and second approximation functions. As indicated above, viscous stresses become significant in the determination of the velocity vector longitudinal component. Fields of remaining parameters can be constructed by taking into account out of all dissipative factors only the heat transfer in the direction of the oncoming stream. In asymmetric perturbations induced by lift of the body the vectors of velocity and thermodynamic quantities equally depend on thermal conductivity and viscosity of gas.

The equation for the longitudinal velocity component is of the form

$$k_5w_{21}\frac{d^2v_x^*}{d\zeta^2}+\left[k_5\left(\frac{dw_{21}}{d\zeta}+\frac{w_{21}}{\zeta}\right)-\rho_{21}(v_{r21}-\zeta)\right]\frac{dv_x^*}{d\zeta}+ \tag{3.2}$$
$$\rho_{21}\left(\frac{2+\varkappa}{2}+ik\frac{\varkappa-1}{2}\right)v_x^* = -k_5\frac{dv_{x21}}{d\zeta}\frac{dw^*}{d\zeta}+\rho_{21}\frac{dv_{x21}}{d\zeta}v_r^* -$$
$$k_5\left(\frac{d^2v_{x21}}{d\zeta^2}+\frac{1}{\zeta}\frac{dv_{x21}}{d\zeta}\right)w^* + \left[(v_{r21}-\zeta)\frac{dv_{x21}}{d\zeta}-\varkappa v_{x21}\right]\rho^*$$

It is separable from system (3.1) and can be integrated after the determination of functions $v_r^*, \ldots, w^*$. The latter satisfy the equations that appear in the analysis of second approximation in the theory of unsteady gas motions. It follows from this that within the accepted accuracy the field of perturbations inside the trail is constructed on the basis of the equivalence principle according to which stream parameters in any $x = \text{const}$ plane are determined independently of the values of its parameters in other planes. The equivalence principle was formulated for perfect (inviscid) flows in [12 – 15].

We eliminate from the second and third of Eqs. (3.1) pressure p^*. Joining to the obtained third order equation the first and fourth of Eqs. (3.1) and using the finite relationship between density ρ^* and specific enthalpy w^*, we form a closed sixth order system for functions v_r^*, v_φ^*, and w^*. The equivalent system for functions with subscripts $2c$ and $2s$ is of the 12-th order with real coefficients. Let us examine the asymptotic behavior of its twelve linearly independent solutions when $\zeta\to\infty$. For brevity we adduce the asymptotics of only one function

$$\rho_{2c} = \zeta^{\alpha_1}[a_1\cos(k_7\ln\zeta)+a_2\sin(k_7\ln\zeta)+O(\zeta^{k_6})]+ \tag{3.3}$$
$$\zeta^{\alpha_2}[a_3\cos(k_8\ln\zeta)+a_4\sin(k_8\ln\zeta)+$$
$$a_5\cos(k_9\ln\zeta)+a_6\sin(k_9\ln\zeta)+O(\zeta^{k_6})]+$$
$$\zeta^{\alpha_3}[a_7\cos(k_7\ln\zeta)+a_8\sin(k_7\ln\zeta)+O(\zeta^{k_6})]+$$
$$\zeta^{\alpha_4}\exp(k_0N_{Pr}\zeta^{-k_6})[a_9\cos(k_{10}\ln\zeta)+a_{10}\sin(k_{10}\ln\zeta)+$$
$$O(\zeta^{k_6})]+\zeta^{\alpha_5}\exp(k_0\zeta^{-k_6})[a_{11}\cos(k_{11}\ln\zeta)+$$
$$a_{12}\sin(k_{11}\ln\zeta)+O(\zeta^{k_6})]$$
$$k_0 = -\frac{\varkappa^2-1}{2\varkappa^2}\frac{k_1^2}{k_2}$$

where the exponents are $\alpha_1 > \alpha_2 > \alpha_3$ and the constants $a_1, \ldots, a_{12}$ are arbitrary. Note that $\alpha_2 = (3 - 2\varkappa) / (\varkappa - 1)$, $k = k_8 \neq k_9$. The dependence of $\alpha_1, \alpha_2, \alpha_3$ and k_7, k_8, and k_9 on $\varkappa$ is clear from Fig. 3. The form of asymptotics of the remaining unknown functions is similar to (3.3).

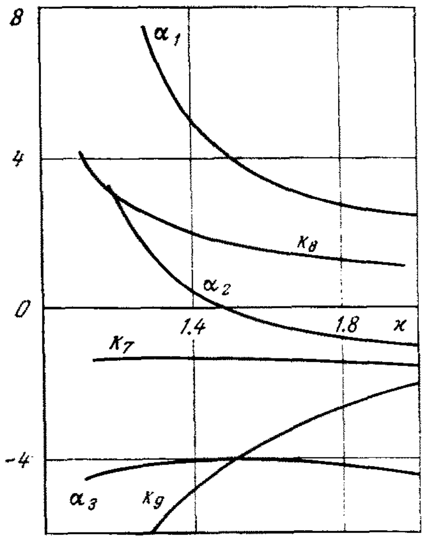

Fig. 3

The asymptotics of all twelve independent solutions have an oscillatory character. The amplitudes of eight of these vary to power laws, while those of the remaining four become exponentially small when $\zeta \to \infty$. Because of this the curves of α_4, α_5 and k_{10}, k_{11} do not appear in Fig. 3. The principal terms of asymptotics with power law amplitudes are determined by the Eulerian part of the system of Eqs. (3.1). The dissipative processes defined by higher derivatives can be neglected when determining these terms. The exponentially small asymptotics are, on the other hand, generated by dissipative factors; the terms in formula (3.5) that are proportional to constants a_9 and a_{10} represent thermal conductivity, while terms with coefficients a_{11} and a_{12} represent viscosity effects.

Comparison of asymptotic expansions (1.7) and (3.3) yields the six constants

$$a_1 = a_2 = 0, \quad a_3 = c_2, \quad a_4 = c_3, \quad a_5 = a_6 = 0 \qquad (3.4)$$

The order of magnitude of the amplitude of asymptotics with constants $a_7, \ldots, a_{12}$ is lower than that of terms in formula (1.7). This implies that no conditions can be imposed on these constants. The third and fourth of equalities (3.4) show that the behavior of second approximation functions makes it possible to join them with the solution for the outer flow region when $\zeta \to \infty$. This possibility is due to the presence in (3.3) of asymptotics whose form is determined by the Eulerian part of Eqs. (3.1).

Let us analyze the behavior of the solution of that system when $\zeta \to 0$. For this we select from the six linearly independent integrals that remain regular when the independent variable infinitely diminishes, the first one which is of the form

$$v_{r2c} = -b_1 \frac{4k_5 k_{12}^2}{c_5 N_{Pr}} + \ldots, \qquad v_{r2s} = b_1 \frac{(\varkappa - 1) k k_{12}}{8c_5 N_{Pr}} \zeta^2 + \ldots \qquad (3.5)$$

$$v_{\varphi 2c} = b_1 \frac{4k_5 k_{12}^2}{c_5 N_{Pr}} \left(1 - \frac{2c_5}{c_4} \zeta^2 + \ldots\right)$$

$$v_{\varphi 2s} = -b_1 \frac{3(\varkappa - 1) k k_{12}}{8c_5 N_{Pr}} \zeta^2 + \ldots$$

$$\rho_{2c} = b_1 \zeta^3 + \ldots, \qquad \rho_{2s} = -b_1 \frac{(\varkappa + 1)^2 k (1 + 2N_{Pr})}{48 \varkappa k_{12}} \zeta^5 + \ldots$$

$$p_{2c} = -\,b_1\,\frac{4\varkappa k_{12}^2}{(\varkappa+1)^2 c_5 N_{Pr}}\left[\frac{(\varkappa+1)\,k_2}{2c_4} - 2\left(\frac{5}{3} - \frac{\lambda_{20}}{\lambda_{10}}\right)k_5 c_5\right]\zeta + \ldots$$

$$p_{2s} = -\,b_1\,\frac{2kk_5k_{12}c_4}{c_5N_{Pr}}\,\zeta + \ldots$$

$$w_{2c} = -\,b_1k_{12}\zeta^3 + \ldots, \quad w_{2s} = b_1\,\frac{(\varkappa+1)^2\,k\,(1+2N_{Pr})}{48\varkappa}\,\zeta^5 + \ldots$$

$$k_{12} = \frac{c_4^2}{k_2}$$

The second regular integral is of the form

$$v_{r2c} = b_2\,\frac{c_4}{48k_2}\left[23 - 5\varkappa + 3\,(\varkappa-1)\,\frac{\lambda_{20}}{\lambda_{10}}\right]\zeta^2 + \ldots \tag{3.6}$$

$$v_{r2s} = b_2\,\frac{kc_4}{16k_2}\left[2 + (\varkappa-1)\left(\frac{5}{3} - \frac{\lambda_{20}}{\lambda_{10}}\right)\right]\zeta^2 + \ldots$$

$$v_{\varphi 2c} = b_2\,\frac{c_4}{16k_2}\left[1 - 3\varkappa - 3\,(\varkappa-1)\,\frac{\lambda_{20}}{\lambda_{10}}\right]\zeta^2 + \ldots$$

$$v_{\varphi 2s} = b_2\,\frac{3kc_4}{16k_2}\left[-2 + (\varkappa-1)\left(1 + \frac{\lambda_{20}}{\lambda_{10}}\right)\right]\zeta^2 + \ldots$$

$$\rho_{2c} = b_2\zeta + \ldots, \quad \rho_{2s} = -\,b_2\,\frac{(\varkappa+1)^2kN_{Pr}}{8\varkappa k_{12}}\,\zeta^3 + \ldots$$

$$p_{2c} = b_2\,\frac{\varkappa k_{12}}{(\varkappa+1)^2\,k_2}\,\zeta + \ldots, \quad p_{2s} = b_2\,\frac{\varkappa k k_{12}}{(\varkappa+1)^2}\,\zeta + \ldots$$

$$w_{2c} = -\,b_2k_{12}\zeta + \ldots, \quad w_{2s} = \frac{(\varkappa+1)^2kN_{Pr}}{8\varkappa}\,\zeta^3 + \ldots$$

We represent the third linearly independent integral by

$$v_{r2c} = -\,b_3\,\frac{12k_5k_{12}^2}{c_5N_{Pr}}\,\zeta^2 + \ldots, \quad v_{r2s} = b_3\,\frac{(\varkappa-1)\,kk_{12}}{4c_5N_{Pr}}\,\zeta^4 + \ldots \tag{3.7}$$

$$v_{\varphi 2c} = b_3\,\frac{36k_5k_{12}^2}{c_5N_{Pr}}\,\zeta^2 + \ldots, \quad v_{\varphi 2s} = -\,b_3\,\frac{5\,(\varkappa-1)\,kk_{12}}{4c_5N_{Pr}}\,\zeta^4 + \ldots$$

$$\rho_{2c} = b_3\zeta^5 + \ldots, \quad \rho_{2s} = -\,b_3\,\frac{(\varkappa+1)^2\,k\,(1+N_{Pr})}{48\varkappa k_{12}}\,\zeta^7 + \ldots$$

$$p_{2c} = -\,b_3\,\frac{48\varkappa^2\,(\varkappa-1)\,k_{12}^2c_4}{(\varkappa+1)^2\,c_5N_{Pr}}\,\zeta + \ldots, \quad p_{2s} = O(\zeta^5)$$

$$w_{2c} = -\,b_3k_{12}\zeta^5 + \ldots, \quad w_{2s} = b_3\,\frac{(\varkappa+1)\,k\,(1+N_{Pr})}{48\varkappa}\,\zeta^7 + \ldots$$

Asymptotic expansion of the other three regular integrals is obtained as follows. First, we substitute in formulas (3.5)–(3.7) the quantities v_{r2s}, $v_{\varphi 2s}$, ρ_{2s}, p_{2s}, w_{2s}, $-v_{r2c}$, $-v_{\varphi 2c}$, $-\rho_{2c}$, $-p_{2c}$ and $-w_{2c}$ for functions v_{r2c}, $v_{\varphi 2c}$, ρ_{2c}, p_{2c}, w_{2c}, v_{r2s}, $v_{\varphi 2s}$, ρ_{2s}, p_{2s} and w_{2s}. The fourth of the sought integrals is then obtained by the substitution in (3.5) of the arbitrary constant b_4 for b_1, and the fifth and sixth regular integrals are obtained by the substitutions in (3.6) of coefficient b_5 for b_2 and in (3.7) of the arbitrary constant b_6 for b_3, respectively.

The remaining six linearly independent solutions have various singularities at zero. Thus in the seventh and eighth solutions density $\rho_{2c} \sim b_7\zeta^{-1}$, $\rho_{2s} \sim b_8\zeta^{-1}$ becomes infinite, in the ninth and tenth pressure $p_{2c} \sim b_9\zeta^{-1}$, $p_{2s} \sim b_{10}\zeta^{-1}$ increases infinitely, and in the eleventh and twelfth solutions the transverse velocity vector components $v_{r2c} \sim v_{\varphi 2c} \sim b_{11}\zeta^{-2}$, $v_{r2s} \sim v_{\varphi 2s} \sim b_{12}\zeta^{-2}$ have singularities. The perturbation field in the outer flow region may become irregular in the vicinity of its inner boundary [5], although in the trail downstream of a streamlined body the excessive values of all gas parameters must remain finite. This statement is equivalent to the equalities

$$b_7 = b_8 = b_9 = b_{10} = b_{11} = b_{12} = 0 \tag{3.8}$$

Let us now formulate the boundary value problem for the system of Eqs. (3.1) as follows: find its solution which for $\zeta \to 0$ satisfies the six conditions (3.8) and for $\zeta \to \infty$ is defined by the asymptotic expansion (3.3) with constants (3.4). Thus one half of boundary conditions for the twelfth order system is specified at one end of the semi-finite interval of integration and the other half at the other end.

The numerical solution of this problem presents serious difficulties because of various reasons. First of all, the coefficients at higher derivatives in the differential equations (3.1) tend rapidly to zero with increasing ζ, because $w_{21} \sim \zeta^{-\varkappa/(\varkappa-1)}$. Second, the presence of exponentially small terms in asymptotic expansions (3.3) means that these equations are to be integrated from 0 to ∞, since integration in the opposite direction is unstable. Third, the specification of boundary conditions at both ends of the semi-infinite interval entails the necessity of adjusting the six coefficients $b_1, \ldots, b_6$ in expansions so as to ensure that when $\zeta \to 0$ the constants $a_1, \ldots, a_6$ in asymptotic formulas have the required values when $\zeta \to \infty$.

We begin the numerical integration of system (3.1) at some point $\zeta \ll 1$, and specify the initial data by the relationships

$$v_{r2c} = \sum_{j=1}^{6} v_{r2c}^{j}(\zeta), \ \ldots, \quad w_{2s} = \sum_{j=1}^{6} w_{2s}^{j}(\zeta) \tag{3.9}$$

where functions $v_{r2c}^{j}, \ldots, w_{2s}^{j}$ are assumed to be six regular asymptotics of the system which are proportional to coefficients b_j. We successively assume only one of these coefficients to be nonzero: $b_1 \neq 0$, $b_2 = b_3 = b_4 = b_5 = b_6 = 0; \ldots;$ $b_1 = b_2 = b_3 = b_4 = b_5 = 0$, $b_6 \neq 0$. Using the computation program for calculating the eight constants $a_1, \ldots, a_8$ in asymptotic formulas for $\zeta \to \infty$, we establish the correspondence between coefficients $b_1, \ldots, b_6$ and the indicated constants. Owing to the linearity of the boundary value problem this correspondence can be represented as

$$(b_1,\ b_2 = b_3 = b_4 = b_5 = b_6 = 0) \searrow (d_{11}b_1, \ \ldots, \ d_{81}b_1); \ldots$$

$$(b_1 = b_2 = b_3 = b_4 = b_5 = 0,\ b_6) \searrow (d_{16}b_6, \ \ldots, \ d_{86}b_6)$$

The numbers $d_{11}, \ldots, d_{86}$ are determined by calculations in which the single nonzero coefficient b_j is made equal unity. To satisfy conditions (3.4) we stipulate that constants $b_1, \ldots, b_6$ must be the solutions of the linear system of algebraic equations

$$\mathbf{Db} = \mathbf{c}, \quad \mathbf{D} = \left\| \begin{matrix} d_{11} & \cdots & d_{16} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ d_{61} & \cdots & d_{66} \end{matrix} \right\|, \quad \mathbf{b} = \left\| \begin{matrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_6 \end{matrix} \right\|, \quad \mathbf{c} = \left\| \begin{matrix} 0 \\ 0 \\ c_2 \\ c_3 \\ 0 \\ 0 \end{matrix} \right\|$$

Constants a_7 and a_8 are determined by formulas

$$a_7 = \sum_{j=1}^{6} d_{7j} b_j, \quad a_8 = \sum_{j=1}^{6} d_{8j} b_j$$

after solving system (3.10) (see below).

The described here method was used for calculating the correction terms which in asymptotics (3.3) are estimated only as regards their order of magnitude, and the computation was terminated at points in the interval $2.5 < \zeta < 3.0$.

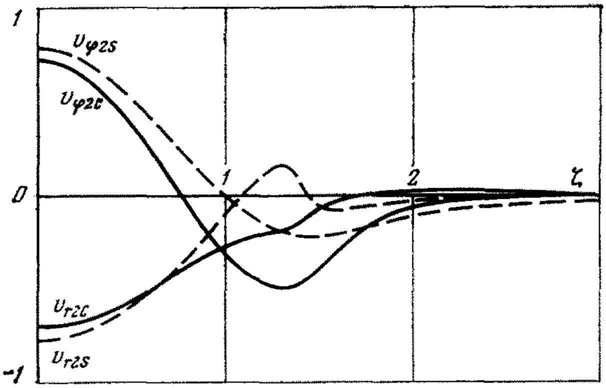

Fig. 4

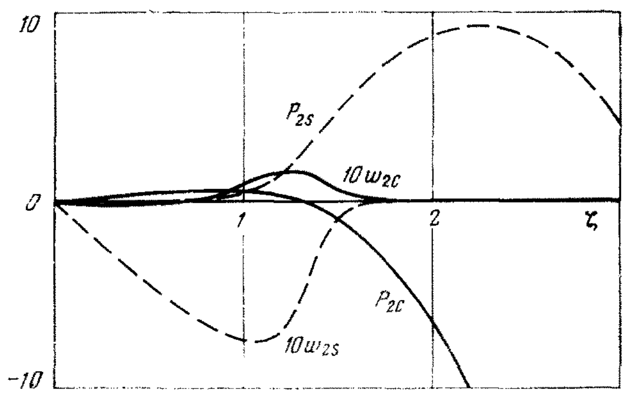

Fig. 5

The properties of gas were specified as follows: $\varkappa = 1.4$, $N_{Pr} = 0.75$, and $\lambda_{20} / \lambda_{10} = 0.1$. The calculations yielded

$$\begin{array}{llll} b_1 = -0.0411, & b_2 = 0.0100, & b_3 = 0.0166, & b_4 = -0.0450 \\ b_5 = 0.1400, & b_6 = 0.0097, & a_7 = -0.3046, & a_8 = -0.7569 \end{array} \qquad (3.10)$$

Repeated integration of system (3.1) with initial data (3.9), into which coefficients (3.10) are substituted, completely solves the boundary value problem. Curves of second approximation functions are shown in Figs. 4 and 5.

It remains to determine the longitudinal component of the stream velocity. Asymptotics of the particular solution of Eq. (3.2) is the same as that specified by formula (1.7) for $\zeta \to \infty$. All four asymptotics for v_{x2c} and v_{x2s} of the corresponding homogeneous equation (3.2) are of an oscillatory character.

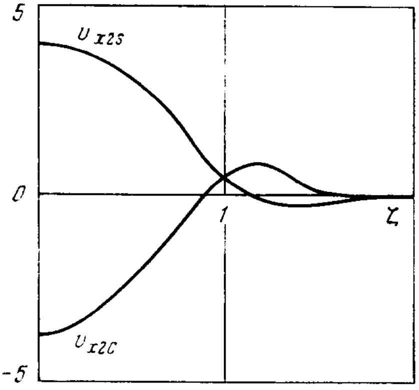

Fig. 6

The amplitudes of two of these tend exponentially to zero, while those of the other two conform to $\zeta^{-\varkappa(2+\varkappa)/(\varkappa-1)}$. As in the determination of function v_{x21}, we define more exactly boundary conditions (1.7) by using higher approximations in the outer region. Estimates show that the arbitrary constants to which are proportional power amplitude asymptotics, are zero. From this we obtain two boundary conditions for $\zeta \to \infty$.

The remaining boundary conditions which must be satisfied when integrating Eq. (3.2) are derived by analyzing the behavior of its solution in the neighborhood of zero where the two linearly independent integrals of the homogeneous equation that corresponds to (3.2) are irregular. The asymptotics of the two regular integrals are simple

$$v_{x2c}^1 = b_{13}\left[1 - \frac{(2+\varkappa)\,k_{12}}{8k_5}\zeta^2 + \ldots\right], \quad v_{x2s}^1 = -\,b_{13}\frac{(\varkappa+1)^2 k k_{12}}{4\varkappa}\zeta^2 + \ldots$$
$$v_{x2c}^2 = b_{14}\frac{(\varkappa+1)^2\,k k_{12}}{4\varkappa}\zeta^2 + \ldots, \quad v_{x2s}^2 = b_{14}\left[1 - \frac{(2+\varkappa)\,k_{12}}{8k_5}\zeta^2 + \ldots\right]$$

The particular integral of the nonhomogeneous equation tends to zero as ζ^3. The stipulation of regularity of solution when $\zeta = 0$ yields the missing boundary conditions which are obtained by equating to zero the coefficients at the two irregular integrals. Curves of functions v_{x2c} and v_{x2s} appear in Fig. 6 for $b_{13} = -3.77$ and $b_{14} = 3.93$.

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